2. THE 'STANDARD TRICK OF MEASURE THEORY'!

While we care about sigma fields only, there are smaller sub-classes that are useful in elucidating the proofs. Here we define some of these.

Definition 11. Let S be a collection of subsets of Ω. We say that S is a

1. **π-system** if $A,B \in S \implies A \cap B \in S$.
2. **λ-system** if (i) $\Omega \in S$. (ii) $A,B \in S$ and $A \subseteq B \implies B \backslash A \in S$. (iii) $A_n \uparrow A$ and $A_n \in S \implies A \in S$.
3. **Algebra** if (i) $\phi, \Omega \in S$. (ii) $A \in S \implies A^c \in S$. (iii) $A,B \in S \implies A \cup B \in S$.
4. **σ-algebra** if (i) $\phi, \Omega \in S$. (ii) $A \in S \implies A^c \in S$. (iii) $A_n \in S \implies \cup A_n \in S$.

We have included the last one again for comparision. Note that the difference between algebras and σ-algebras is just that the latter is closed under countable unions while the former is closed only under finite unions. As with σ-algebras, arbitrary intersections of algebras/λ-systems/π-systems are again algebras/λ-systems/π-systems and hence one can talk of the algebra generated by a collection of subsets etc.

Example 12. The table below exhibits some examples.

Ω	S ($\pi-$system)	$\mathcal{A}(S)$ (algebra generated by S)	$\sigma(S)$
$(0,1]$	$\{(a,b] \ : \ 0 < a \le b \le 1\}$	$\{\cup_{k=1}^{N}(a_k,b_k] \ : \ 0 < a_1 \le b_1 \le a_2 \le b_2 \ldots \le b_N \le 1\}$	$\mathcal{B}(0,1]$
$[0,1]$	$\{(a,b] \cap [0,1] \ : \ a \le b\}$	$\{\cup_{k=1}^{N} R_k \ : \ R_k \in S$ are pairwise disjoint$\}$	$\mathcal{B}[0,1]$
$\mathbb{R}^d$	$\{\prod_{i=1}^{d}(a_i,b_i] \ : \ a_i \le b_i\}$	$\{\cup_{k=1}^{N} R_k \ : \ R_k \in S$ are pairwise disjoint$\}$	$\mathcal{B}_{\mathbb{R}^d}$
$\{0,1\}^{\mathbb{N}}$	collection of all cylinder sets	finite disjoint unions of cylinders	$\mathcal{B}(\{0,1\}^{\mathbb{N}})$

Often, as in these examples, sets in a π-system and in the algebra generated by the π-system can be described explicitly, but not so the sets in the generated σ-algebra.

Clearly, a σ-algebra is an algebra is a π-systemas well as a λ-system. The following converse will be useful. Plus, the proof exhibits a basic trick of measure theory!

Lemma 13 (Sierpinski-Dynkin $\pi-\lambda$ theorem)**.** *Let Ω be a set and let $\mathcal{F}$ be a set of subsets of Ω.*

1. *$\mathcal{F}$ is a σ-algebra if and only if it is a π-system as well as a λ-system.*
2. *If S is a π-system, then $\lambda(S) = \sigma(S)$.*

Proof. (1) One way is clear. For the other way, suppose $\mathcal{F}$ is a π-system as well as a λ-system. Then, $\phi, \Omega \in \mathcal{F}$. If $A \in \mathcal{F}$, then $A^c = \Omega \backslash A \in \mathcal{F}$. If $A_n \in \mathcal{F}$, then the finite unions $B_n := \cup_{k=1}^{n} A_k = \left(\cap_{k=1}^{n} A_k^c\right)^c$ belong to $\mathcal{F}$ as $\mathcal{F}$ is a π-system. The countable union $\cup A_n$ is the increasing limit of B_n and hence belongs to $\mathcal{F}$ by the λ-property.

(2) By part (i), it suffices to show that $\mathcal{F} := \lambda(S)$ is a π-system, that is, we only need show that if $A,B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. This is the tricky part of the proof!

Fix $A \in S$ and let $\mathcal{F}_A := \{B \in \mathcal{F} \ : \ B \cap A \in \mathcal{F}\}$. S is a π-system, hence $\mathcal{F}_A \supset S$. We claim that $\mathcal{F}_A$ is a λ-system. Clearly, $\Omega \in \mathcal{F}_A$. If $B,C \in \mathcal{F}_A$ and $B \subset C$, then $(C \backslash B) \cap A = (C \cap A) \backslash (B \cap A) \in \mathcal{F}$ because $\mathcal{F}$ is a λ-system containing $C \cap A$ and $B \cap A$. Thus $(C \backslash B) \in \mathcal{F}_A$. Lastly, if $B_n \in \mathcal{F}_A$ and $B_n \uparrow B$, then $B_n \cap A \in \mathcal{F}_A$ and $B_n \cap A \uparrow B \cap A$. Thus $B \in \mathcal{F}_A$. This means that $\mathcal{F}_A$ is a λ-system containing S and hence $\mathcal{F}_A \supset \mathcal{F}$. In other words, $A \cap B \in \mathcal{F}$ for all $A \in S$ and all $B \in \mathcal{F}$.

Now fix any $A \in \mathcal{F}$. And again define $\mathcal{F}_A := \{B \in \mathcal{F} \ : \ B \cap A \in \mathcal{F}\}$. *Because of what we have already shown*, $\mathcal{F}_A \supset S$. Show by the same arguments that $\mathcal{F}_A$ is a λ-system and conclude that $\mathcal{F}_A = \mathcal{F}$ for all $A \in \mathcal{F}$. This is another way of saying that $\mathcal{F}$ is a π-system. ■

As an application, we prove a certain uniqueness of extension of measures.

Lemma 14. *Let S be a π-system of subsets of Ω and let $\mathcal{F} = \sigma(S)$. If $\mathbf{P}$ and $\mathbf{Q}$ are two probability measures on $\mathcal{F}$ such that $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in S$, then $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $A \in \mathcal{F}$.*

Proof. Let $T = \{A \in \mathcal{F} \ : \ \mathbf{P}(A) = \mathbf{Q}(A)\}$. By the hypothesis $T \supset S$. We claim that T is a λ-system. Clearly, $\Omega \in T$. If $A,B \in T$ and $A \supset B$, then $\mathbf{P}(A \backslash B) = \mathbf{P}(A) - \mathbf{P}(B) = \mathbf{Q}(A) - \mathbf{Q}(B) = \mathbf{Q}(A \backslash B)$ implying that $A \backslash B \in T$. Lastly, if $A_n \in T$ and $A_n \uparrow A$, then $\mathbf{P}(A) = \lim_{n\to\infty} \mathbf{P}(A_n) = \lim_{n\to\infty} \mathbf{Q}(A_n) = \mathbf{Q}(A)$. Thus $T \supset \lambda(S)$ which is equal to $\sigma(S)$ by Dynkin's $\pi - \lambda$ theorem. Thus $\mathbf{P} = \mathbf{Q}$ on $\mathcal{F}$. ■

Theorem 15. *There exists a unique Borel probability measure* $\mathbf{m}$ *on* $[0,1]$ *such that* $\mathbf{m}(I) = |I|$ *for any interval* I.

[**Sketch of the proof**] Note that $S = \{(a,b] \cap [0,1]\}$ is a π-system that generate $\mathcal{B}$. Therefore by Lemma 14, uniqueness follows. Existence is all we need to show. There are two steps.

Step 1 - Construction of the outer measure $\mathbf{m}_*$ Recall that we define $\mathbf{m}_*(A)$ for any subset by

$$\mathbf{m}_*(A) = \inf\left\{\sum_k |I_k| \ : \ \text{each } I_k \text{ is an open interval and } \{I_k\} \text{ a countable cover for } A\right\}.$$

$\mathbf{m}_*$ has the following properties. (i) $\mathbf{m}_*$ is a $[0,1]$-valued function defined on all subsets $A \subset \Omega$. (ii) $\mathbf{m}_*(A \cup B) \leq \mathbf{m}_*(A) + \mathbf{m}_*(B)$ for any $A, B \subset \Omega$. (iii) $\mathbf{m}_*(\Omega) = 1$.

These properties constitute the definition of an **outer measure**. In the case at hand, the last property follows from the following exercise.

Exercise 16. Show that $\mathbf{m}_*(a,b] = b - a$ if $0 < a \leq b \leq 1$.

Clearly, we also get countable subadditivity $\mathbf{m}_*(\cup A_n) \leq \sum \mathbf{m}_*(A_n)$. The difference from a measure is that equality might not hold, even if the sets are pairwise disjoint.

Step-2 - The σ-field on which $\mathbf{m}_*$ is a measure

Let $\mathbf{m}_*$ be an outer measure on a set Ω. Then by restricting $\mathbf{m}_*$ to an appropriate σ-fields one gets a measure. We would also like this σ-field to be large (not the sigma algebra $\{\emptyset, \Omega\}$ please!).

Cartheodary's brilliant definition is to set

$$\mathcal{F} := \{A \subset \Omega \ : \ \mathbf{m}_*(E) = \mathbf{m}_*(A \cap E) + \mathbf{m}_*(A^c \cap E) \text{ for any } E\}.$$

Note that subadditivity implies $\mathbf{m}_*(E) \leq \mathbf{m}_*(A \cap E) + \mathbf{m}_*(A^c \cap E)$ for any E for any A, E. The non-trivial inequality is the other way.

Theorem 17. *Then,* $\mathcal{F}$ *is a sigma algebra and* μ_* *restricted to* $\mathcal{F}$ *is a p.m.*

Proof. It is clear that $\emptyset, \Omega \in \mathcal{F}$ and $A \in \mathcal{F}$ implies $A^c \in \mathcal{F}$. Next, suppose $A, B \in \mathcal{F}$. Then for any E,

$$\mathbf{m}_*(E) = \mathbf{m}_*(E \cap A) + \mathbf{m}_*(E \cap A^c) = \mathbf{m}_*(E \cap A \cap B) + \{\mathbf{m}_*(E \cap A \cap B^c) + \mathbf{m}_*(E \cap A^c)\} \geq \mathbf{m}_*(E \cap A \cap B) + \mathbf{m}_*(E \cap (A \cap B)^c))$$

where the last inequality holds by subadditivity of $\mathbf{m}_*$ and $(E \cap A \cap B^c) \cup (E \cap A^c) = E \cap (A \cap B)^c$. Hence $\mathcal{F}$ is a π-system.

As $A \cup B = (A^c \cap B^c)^c$, it also follows that $\mathcal{F}$ is an algebra. For future use, note that $\mathbf{m}_*(A \cup B) = \mathbf{m}_*(A) + \mathbf{m}_*(B)$ if A, B are disjoint sets in $\mathcal{F}$. To see this apply the definition of $A \in \mathcal{F}$ with $E = A \cup B$.

It suffices to show that $\mathcal{F}$ is a λ-system. Suppose $A, B \in \mathcal{F}$ and $A \supset B$. Then

$$\mathbf{m}_*(E) = \mathbf{m}_*(E \cap B^c) + \mathbf{m}_*(E \cap B) = \mathbf{m}_*(E \cap B^c \cap A) + \mathbf{m}_*(E \cap B^c \cap A^c) + \mathbf{m}_*(E \cap B) \geq \mathbf{m}_*(E \cap (A \backslash B)) + \mathbf{m}_*(E \cap (A \backslash B)^c).$$

Before showing closure under increasing limits, Next suppose $A_n \in \mathcal{F}$ and $A_n \uparrow A$. Then $\mathbf{m}_*(A) \geq \mathbf{m}_*(A_n) = \sum_{k=1}^n \mathbf{m}_*(A_k \backslash A_{k-1})$ by finite additivity of $\mathbf{m}_*$. Hence $\mathbf{m}_*(A) \geq \sum \mathbf{m}_*(A_k \backslash A_{k-1})$. The other way inequality follows by subadditivity of $\mathbf{m}_*$ and we get $\mathbf{m}_*(A) = \sum \mathbf{m}_*(A_k \backslash A_{k-1})$. Then for any E we get

$$\mathbf{m}_*(E) = \mathbf{m}_*(E \cap A_n) + \mathbf{m}_*(E \cap A_n^c) \geq \mathbf{m}_*(E \cap A_n) + \mathbf{m}_*(E \cap A^c) = \sum_{k=1}^n \mathbf{m}_*(E \cap (A_k \backslash A_{k-1})) + \mathbf{m}_*(E \cap A^c).$$

The last equality follows by finite additivity of $\mathbf{m}_*$ on $\mathcal{F}$. Let $n \to \infty$ and use subadditivity to see that

$$\mathbf{m}_*(E) \geq \sum_{k=1}^\infty \mathbf{m}_*(E \cap (A_k \backslash A_{k-1})) + \mathbf{m}_*(E \cap A^c) \geq \mathbf{m}_*(E \cap A) + \mathbf{m}_*(E \cap A^c).$$

Thus, $A \in \mathcal{F}$ and it follows that $\mathcal{F}$ is a λ-system too and hence a σ-algebra.

Lastly, if $A_n \in \mathcal{F}$ are pairwise disjoint with union A, then $\mathbf{m}_*(A) \geq \mathbf{m}_*(A_n) = \sum_{k=1}^n \mathbf{m}_*(A_k) \to \sum_k \mathbf{m}_*(A_k)$ while the other way inequality follows by subadditivity of $\mathbf{m}_*$ and we see that $\mathbf{m}_*|_{\mathcal{F}}$ is a measure.

Step-3 - $\mathcal{F}$ is large enough!

Let $A=(a,b]$. For any $E\subset[0,1]$, let $\{I_n\}$ be an open cover such that $\mathbf{m}_*(E)\geq\sum|I_n|$. Then, note that $\{I_n\cap(a,b)\}$ and $\{I_n\cap[a,b]^c\}$ are open covers for $A\cap E$ and $A^c\cap E$, respectively ($I_n\cap[a,b]^c$ may be a union of two intervals, but that does not change anything essential). It is also clear that $|I_n|=|I_n\cap(a,b)|+|I_n\cap(a,b)^c|$. Hence we get

$$\mathbf{m}_*(E)\geq\sum|I_n\cap(a,b)|+\sum|I_n\cap(a,b)^c|\geq\mathbf{m}_*(A\cap E)+\mathbf{m}_*(A^c\cap E).$$

The other inequality follows by subadditivity and we see that $A\in\mathcal{F}$. Since the intervals $(a,b]$ generate $\mathcal{B}$, and $\mathcal{F}$ is a sigma algebra, we get $\mathcal{F}\supset\mathcal{B}$. Thus, restricted to $\mathcal{B}$ also, $\mathbf{m}_*$ gives a p.m. ■

Remark 18. (1) We got a σ-algebra $\mathcal{F}$ that is larger than $\mathcal{B}$. Two natural questions. Does $\mathcal{F}$ or $\mathcal{B}$ contain all subsets of $[0,1]$? Is $\mathcal{F}$ strictly larger than $\mathcal{B}$? We show that $\mathcal{F}$ does not contain all subsets. One of the homework problems deals with the relationship between $\mathcal{B}$ and $\mathcal{F}$.

(2) $\mathbf{m}$, called the Lebesgue measure on $[0,1]$, is the only probability space one ever needs. In fact, all probabilities ever calculated can be seen, in principle, as calculating the Lebsgue measure of some Borel subset of $[0,1]$!

Generalities The construction of Lebesgue measure can be made into a general procedure for constructing interesting measures, starting from measures of some rich enough class of sets. The steps are as follows.

(1) Given an algebra $\mathcal{A}$ (in this case finite unions of $(a,b]$), and a *countably additive p.m* $\mathbf{P}$ on $\mathcal{A}$, define an outer measure $\mathbf{P}_*$ on all subsets by taking infimum over countable covers by sets in $\mathcal{A}$.
(2) Then define $\mathcal{F}$ exactly as above, and prove that $\mathcal{F}\supset\mathcal{A}$ is a σ-algebra and $\mathbf{P}_*$ is a p.m. on $\mathcal{A}$.
(3) Show that $\mathbf{P}_*=\mathbf{P}$ on $\mathcal{A}$.

Proofs are quite the same. Except, in $[0,1]$ we started with $\mathbf{m}$ defined on a π-system S rather than an algebra. But in this case the generated algebra consists precisely of disjoint unions of sets in S, and hence we knew how to define $\mathbf{m}$ on $\mathcal{A}(S)$. When can we start with $\mathbf{P}$ defined ona π-system? The crucial point in $[0,1]$ was that for any $A\in S$, one can write A^c as a finite union of sets in S. In such cases (which includes examples from the previous lecture) the generated algebra is precisely the set of disjoint finite unions of sets in S and we define $\mathbf{P}$ on $\mathcal{A}(S)$ and then proceed to step one above.

Exercise 19. Use the general procedure as described here, to construct the following measures.

(a) A p.m. on $([0,1]^d,\mathcal{B})$ such that $\mathbf{P}([a_1,b_1]\times\ldots\times[a_d,b_d])=\prod_{k=1}^d(b_k-a_k)$ for all cubes contained in $[0,1]^d$. This is the d-dimensional Lebesgue measure.

(b) A p.m. on $\{0,1\}^{\mathbb{N}}$ such that for any cylinder set $A=\{\omega\,:\,\omega_{k_j}=\varepsilon_j,\ j=1,\ldots,n\}$ (any $n\geq1$ and $k_j\in\mathbb{N}$ and $\varepsilon_j\in\{0,1\}$) we have (for a fixed $p\in[0,1]$ and $q=1-p$)

$$\mathbf{P}(A)=\prod_{j=1}^{n}p^{\varepsilon_j}q^{1-\varepsilon_j}.$$

[**Hint:** Start with the algebra generated by cylinder sets].